Towards the integral: from area estimation to the rectangle method

The activity aims to take the first steps towards the concept of integrals (without introducing it explicitly) by following a path that takes measurement as its fundamental starting point, in particular the measurement of areas under functions. In developing the activity, reference was made to three crucial aspects of educational research:

- the transition of students from intuition and perception to conceptualisation, in terms of cognitive processes (Sfard, 1991; Dubinsky, 1991; Tall, 2000);
- the mediation offered by technology in this context (Verillon & Rabardel, 1995);
- the teacher's intervention in the construction of knowledge in a social context, i.e. during the “mathematical discussion” (Bartolini Bussi et al., 1999).

In particular, it is important to note that the area under the function graph is a *cognitive root* (Tall, 2002) for the concept of integral, in that:

- it is cognitively meaningful for students at the beginning of the learning sequence. This is not the case for the concept of primitives as the inverse of derivatives, which is “artificially” linked to definite integrals through a theorem, the fundamental theorem of integral calculus;
- it allows to start with perceptual aspects, which are the most accessible to the student,
- it allows for extensive exploration in numerical and graphical terms, before constructing the *meaning* of the concept in terms of process, then studying its properties as a mathematical object;
- it is a concept that is consistent enough to remain useful even when developing a more sophisticated understanding of integral calculus..

The proposal is divided into several stages: the first stages are carried out using pen and paper, and can also be completed in grades 9-10, while the last two stages are more advanced and make use of dynamic geometry software, which mediates the more “calculative” aspect of the procedure, which is to be delegated to the tool.

Task	Type of task	Materials and tools
1. Area of an outline	**Group work + collective discussion** Determine the area of a given outline.	paper and pencil (ruler)
2. Area with grid lines	**Group work + collective discussion** Use graph paper to determine the area of the shape.	paper and pencil (ruler, sheets with different grid patterns)
3. Area under the curve	**Group work + collective discussion** How can we determine the area under the given curve?	Paper and pencil + GeoGebra
4. Example from physics	**Group work + collective discussion** Isothermal transformation.	Paper and pencil + GeoGebra

Task 1

Students are asked to make a rough shape (e.g. from the outline of their hand) or one can be provided to all groups (Fig. 1). They are then asked to determine the area of the shape, choosing one or more methods, and then explain which method(s) they used and why. The shape should be sufficiently irregular so that triangulation alone cannot be used, but rather “challenges” the group of students to seek methods of approximation.

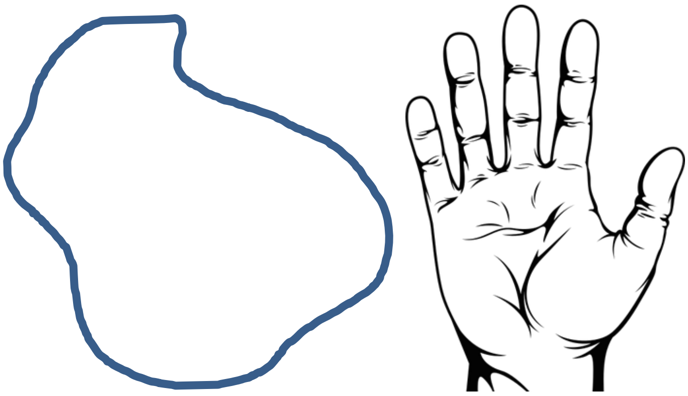


Figure 1. Examples of shapes for which the area must be

Questions for the teachers:

- What methods and tools do you expect students to use? Why?
- Which method in particular do you think would be interesting to address collectively during the class discussion?

Questions for guiding the students' discussion:

- What is the mathematical validity of observations made with measurement experience?
- What competences did you use to carry out the activity?
- Imagine you could develop a tool that uses the method(s) you applied to calculate the area of any shape. What would it look like? What aspects would you need to consider in order to use it with any shape?

Task 2

It is then proposed to work with different types of graph paper[1] to compare the area values obtained (by excess and by defect) in the various cases.

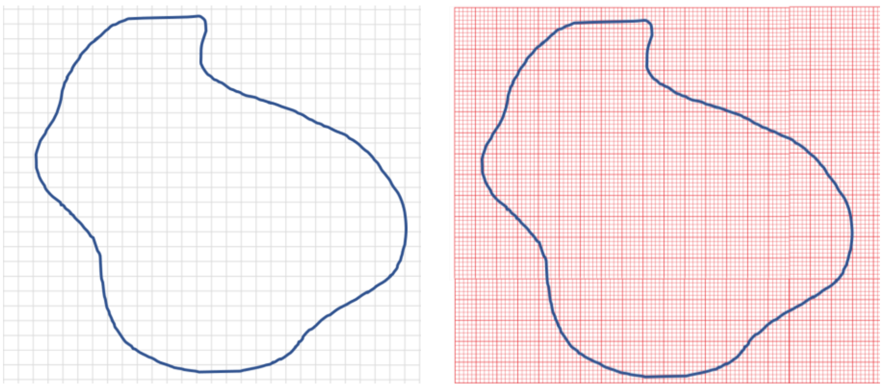


Figura 2. Confronto di quadrettature

[1] https://museprintables.com/c/paper/ for printable formats of graph paper.

Questions for the teachers:

- What critical aspects related to the concept of measurement do you expect to emerge in this activity?

This activity could also be carried out in GeoGebra (for example, during the group discussion phase) using the Zoom and the grid provided to "instantly" modify the reference grid (however, it is more difficult to set the square count as the scale changes). Conceptually, it is a powerful tool that can also be used as a first analogy with the transition to the limit and the density of rational numbers.

Questions for guiding the students' discussion:

- How can the accuracy of the solution be increased?
- How can the exact value of the area be obtained?

Task 3

The teacher introduces the rectangle method. Students work in groups to explore how to calculate the area under a parabola in GeoGebra (https://www.geogebra.org/m/cmadp3jp).

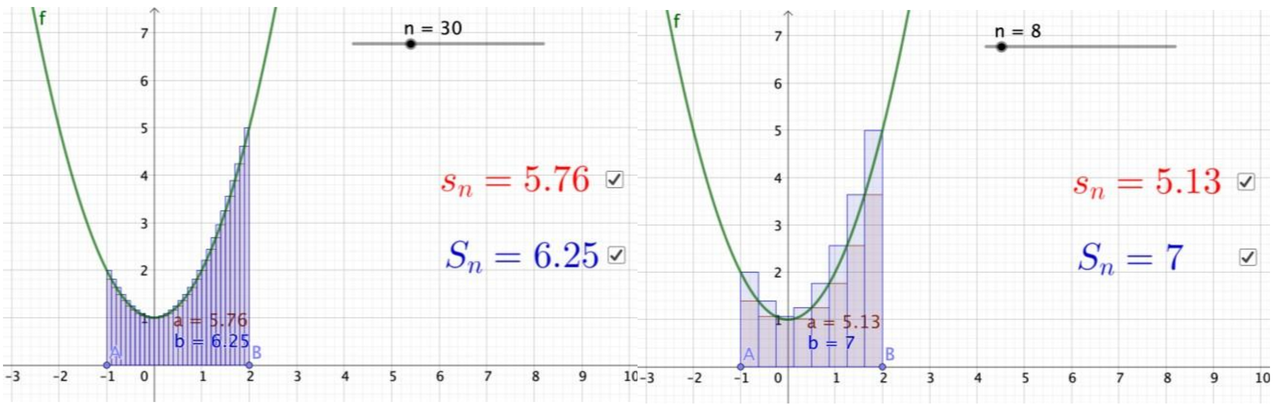


Figure 3. Lower and upper sums as the number of rectangles varies in GeoGebra

Questions for the teachers:

- A crucial cognitive aspect of this activity is the transition to the concept in terms of process (the approximation of an area under a curve using finite sums), which potentially continues indefinitely (as the number of rectangles increases). What role does dynamic geometry software play in this transition?

Questions for guiding the students' discussion:

- What happens as the number of rectangles increases?
- What happens to the two sequences as the number of rectangles increases?

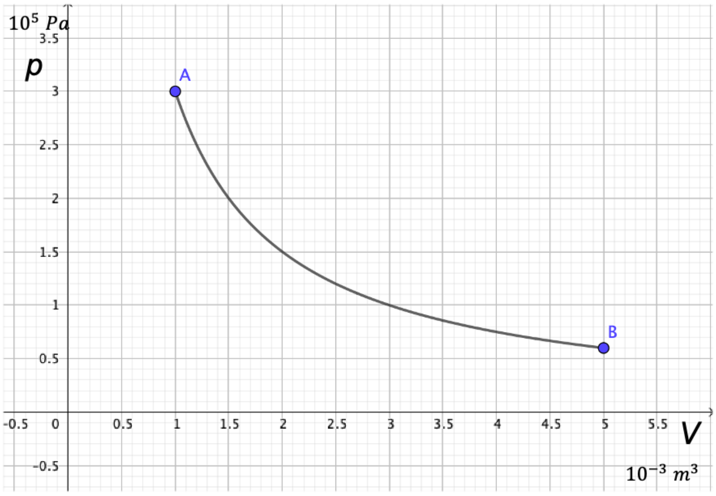
Task 4

We propose contextualising the determination of area in a physical context (thermodynamics) as an example, in which area takes on the meaning of work done by a system. Similar tasks can also be carried out in the context of dynamics (more suitable for a Year 9 class), for example in the case of the hourly laws of uniformly accelerated motion.

Work of an isothermal transformation

Students are asked to work on the following situation.

The figure shows an isothermal transformation of a system from initial state A to final state B in a pressure-volume diagram (Clapeyron plane):



1. Write the equation of the isothermal transformation shown in the graph.
2. Does the system perform or undergo work? Justify your answer.
3. Determine this work, expressing the result in an appropriate unit of measurement.
4. Explore the problem in GeoGebra (https://www.geogebra.org/m/tbkcumes).

References

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