MISTER X: INQUIRY IN ALGEBRA

This activity is developed, in line with the Variation Method, by proposing a series of stimuli that aim to explore relationships using algebraic language. These are activities that encourage argumentation and demonstration using the symbolic language of natural numbers obtained from the four operations (mainly addition and multiplication).

From UMI *Matematica 2001*:
"The core process of ARGUMENTING AND CONJECTURING characterises the activities that prepare students for demonstration, which is one of the activities that distinguish mature mathematical thinking, as will be acquired in […] school."
"Argumentative activity is like a discourse that allows the subject to return to what has been done, producing interpretations, explanations, answers to questions such as 'why is this so?'"

***Preliminary reflections:* Write and add even and odd numbers, and reasoning.**
From INVALSI (similar to OCSE-PISA):
Question for grade 8:
Is the sum of a natural number n and its successor $n + 1$ always an odd number? Choose one of the two answers and complete the sentence.

- ☐ Yes, because…
- ☐ No, because…

Questions for the teachers:
What is the purpose of this question? *What reasoning would you consider correct?*

From the results: 58.2% of students in the statistical sample answered this question incorrectly, demonstrating widespread difficulty both in algebraic manipulation and the relative attribution of meanings, and in terms of reasoning. We note that in this type of question, numerical exploration is fundamental but not sufficient for generalisation. While we can falsify a statement using a counterexample, showing that it is true in certain specific cases is not sufficient to prove its validity. Discussing this tension between the need for examples to explore the situation mathematically and their insufficiency for proof is a key point for teaching.

Another idea:
The sum of two natural numbers a and b is even. The product of a and b is odd. The numbers a and b are:

- ☐ Both even
- ☐ Both odd
- ☐ One even one odd
- ☐ We cannot say

Justify your answer by explaining your reasoning.

Questions for the teachers:
What is the purpose of this question? *What reasoning would you consider correct?*

Proposals for classroom work

(1)

- If b is an odd number, what can you say about the number $2b$? And about the number $3b$? Are they also odd? Explain why.
- If you multiply b by 100, do you still get an odd number? What if you multiply it by 101? Explain why.
- Now try to generalise what you have found. How can you express in general terms the properties you have conjectured in the previous points?
- How can you prove them?

(2)

- Consider the product of two consecutive natural numbers: what can you say about the result?
- What if there were three numbers? What if there were four? What if there were five?
- Generalise and explain why the properties you have conjectured are valid.

- What if, instead of consecutive natural numbers, we considered consecutive even numbers?
- What if they were consecutive odd numbers?
- Generalise and explain why the properties you have conjectured are valid.

(3)

- Consider the sum of two consecutive natural numbers: what can you say about the result?
- What if there were three numbers? What if there were four? What if there were five?
- Generalise and explain why the properties you have conjectured are valid.
- What if, instead of consecutive natural numbers, we considered consecutive even numbers?
- What if they were consecutive odd numbers?
- What if they were consecutive multiples of 3?
- What if they were consecutive multiples of 4?
- Can you write down the properties that generalise these situations?

(4)

- Consider a natural number. Determine the square of its successor and the square of its predecessor. Subtract the two squares: what can you say about their difference?
- Repeat the process with other numbers. Are there any patterns you have found in all the examples you have seen? Justify them.
- How would you express these patterns in general? Try to prove them.

Task for the teachers
1. Solve the tasks.
2. Discuss them observing specifically the structure of the questions.
3. Add one or more question(s) to the previous tasks.
4. Make up a new task.

Ideas for further reflection

1. Healy & Hoyles (2000) presented a questionnaire to a sample of English students aged between 14 and 15. One of the questions asked them to consider various attempts at proof proposed by students to demonstrate the validity of the following statement:

When you multiply three consecutive natural numbers, the result is always a multiple of 6.

KATE
A multiple of 6 must have factors 3 and 2.
If you have three consecutive numbers, one will be a multiple of 3 because every three numbers one is in the three times table.
Also, at least one number will be even because all even numbers are multiples of 2.
If I multiply three consecutive numbers, the result must have at least one factor 3 and one factor 2.
So Kate concludes that it is true.

LEON	MARIA
$1 \times 2 \times 3 = 6$ $3 \times 3 \times 4 = 24$ $4 \times 5 \times 6 = 120$ $6 \times 7 \times 8 = 336$ So Leon concludes that it is true.	x is a natural number $x(x+1)(x+2)=$ $=(x^2+x)(x+2)=$ $=x^3+ x^2+2x+2x$ Canceling the x we have: 1+1+2+2=6 So Maria concludes that it is true.

NISHA
Of three consecutive numbers, the first is
EVEN, and can be written as 2a (a is any natural number) or
ODD, and can be written as 2b-1 (b is any integer).

If EVEN
$2a(2a+1)(2a+ 2)$ is multiple of 2
and a is either multiple of 3 DONE
or a is not a multiple of 3,
therefore 2a is not a multiple of 3
but $(2a+1)$ is a multiple of 3 or $(2a+2)$ is a multiple of 3
DONE

If ODD
$(2b-1)(2b)(2b+1)$ is multiple of 2
And b is multiple of 3 DONE
or b is not multiple of 3
therefore 2b is not a multiple of 3
but $(2b-1)$ is multiple of 3 or $(2b+1)$ is multiple of 3
DONE

Next, students must choose, from the proposed proofs: (a) the one closest to the proof they themselves would have proposed as a solution, and (b) the one that, in their opinion, would have received the best mark from the teacher.

Question for the teachers
Examine the proposed proofs. Which ones would your students produce? What types of proof do you think they would consider most appreciated by the teacher? Why?

2. An extract from Osama Swidan, Annalisa Cusi, Ornella Robutti, and Ferdinando Arzarello's paper on the Method of Varying Inquiry (MVI):

Method of Varying Inquiry

MVI consists of designing activities as challenging tasks for students (as in the inquiry approach), in meaningful contexts (real-world or mathematical), by varying some variables of phenomena while keeping the others invariant (as in variation theory) to let students discern the object of learning embedded in the phenomena. Moreover, to create a model that could operationally support the teachers in fostering and leading inquiry processes in their classes, we add two elements useful to a meticulous design of classroom activities and discussions that foster students' inquiry processes.
The first element is didactical: the *mathematics laboratory*, elaborated in the institutional context of the Italian Ministry of Education (Anichini et al., 2003) and representing a teaching approach based on group and peer work, sharing and comparing ideas, classroom discussions led by the teacher, and "acting" instead of "listening" through problem posing and problem solving. It is aimed at fostering the construction of meanings of mathematical objects using different tools and through social interaction. The inspiring idea was, in fact, that of the Renaissance-era workshop, in which the apprentices learned by doing, seeing, imitating, and communicating with each other.
The second element is theoretical: the *virtuous cycle* (see Figure 1), introduced by Swidan & colleagues (2023) as a process that supports students in making sense of mathematics by enabling them to connect different pieces of theoretical knowledge (not just mathematical). The cycle draws its origins from similar and more complex cycles for using formal mathematics to interpret real-world situations.

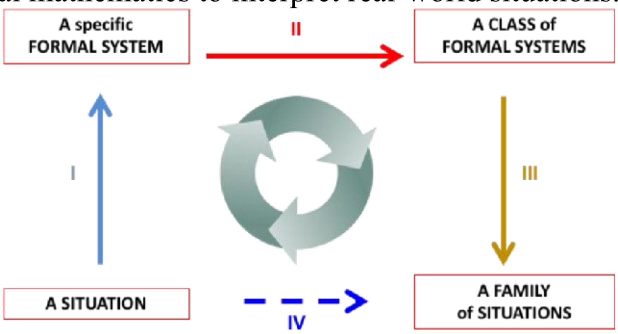

Fig. 1 Virtuous cycle (adapted from Swidan et al., 2023)

The *virtuous cycle* consists of four intertwined processes: (*I*) representation of aspects of a situation (related to a phenomenon in a real-world or mathematical context) into a formal system; (*II*) treatment of the representation within a formal system or conversions between systems towards a generalization in a class of formal systems;[1] (*III*) interpretation of the generalisation in relation to a family of situations; (*IV*) interpretation of the initial situation within a family of situations. It is important to stress that (IV) is, therefore, the outcome of (I), (II), and (III). We take the notion of situation (or problem-situation) as a primitive idea. A situation could refer both to real-world or mathematical phenomena and it represents for us a macro-object of inquiry. This object becomes a problem when questions are posed, making the students focus on specific variables that emerge when phenomena are analysed and on the relationships between these variables.

[1] The terminology employed here is from Duval. *Treatment* refers to transformations within the same register, e.g., when $(x-1)^2+(y-1)^2=2$ is developed into $x^2+y^2-2x-2y+2=0$. *Conversion* refers to transformations between different registers, e.g., when the previous formula is interpreted as a circle in the Cartesian plane.

When we use the term “formal system” we mean not only the register that is chosen to construct representations, but also all the theoretical frames (properties used, typical procedures, …) within which the situation is represented, interpreted, and studied. Of course, the cycle is productive if arrow IV (in Fig. 1) represents a genuine “epistemic gain” in how the students interpret the initial situation after the cycle, that is, if they consider the explored situation as particular case of a family of situations.
The MVI-model interprets learning as a layered inquiry process (see Fig.2) promoted - in the context of the mathematics laboratory – through a task-design that may support students in making sense of mathematical concepts through the processes that characterize the virtuous cycle.

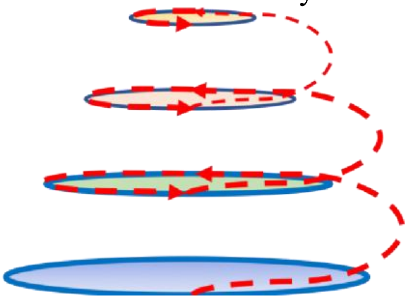

Fig. 2 MVI: a spiral-shaped and multi-layered model

This layered process to the building of meanings is a novel element introduced by the MVI-model with respect to the inquiry approach and to variation theory. The several levels of inquiry that characterize the MVI are represented in Figure 3.

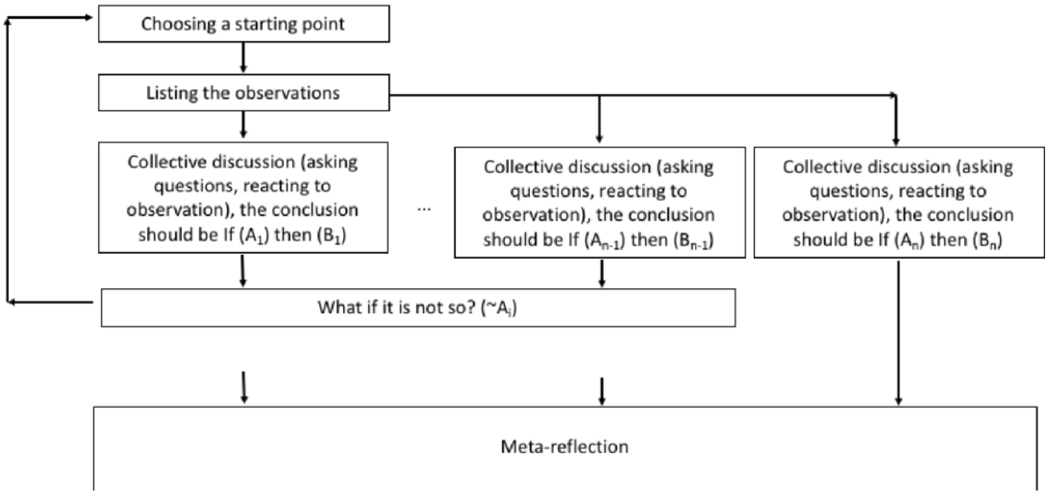

Fig. 3 Levels of inquiry in the MVI

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